A Livšic type theorem for germs of analytic diffeomorphisms

Andrés Navas & Mario Ponce

Abstract. We deal with the problem of the validity of Livšic's theorem for cocycles of diffeomorphisms satisfying the orbit periodic obstruction over an hyperbolic dynamics. We give a result in the positive direction for cocycles of germs of analytic diffeomorphisms at the origin.

1 Introduction

Given a map (dynamical system) $T : X \to X$ over a compact metric space X and a (topological) group $\mathcal{G}$, we consider a continuous $\mathcal{G}$-valued cocycle $A\colon \mathbb{N} \times X \to \mathcal{G}$, that is, a continuous map taking values in $\mathcal{G}$ satisfying the cocycle relation

$$A(n+m,x) = A(n,T^m x)A(m,x)$$

for every m,n in $\mathbb{N}$ and every $x \in X$. This cocycle is completely determined by the continuous function $A(\cdot) := A(1,\cdot) : X \to \mathcal{G}$, and the cocycle relation yields

$$A(n,x) = A(T^{n-1}x)A(T^{n-2}x)\cdots A(x)$$

for every $n \geq 1$. A natural problem is to determine conditions ensuring for such a cocycle to be conjugated to a cocycle taking values in a "small" subgroup of $\mathcal{G}$. For the case of the trivial subgroup, this property means that there exists a continuous function $B\colon X \to \mathcal{G}$ such that

$$A(x) = B(Tx)B(x)^{-1} \quad \text{for all} \quad x \in X. \tag{1}$$

Whenever this *cohomological equation* associated to the cocycle A has a solution B, we say that A is a *coboundary*. The simplest obstruction for the existence of B is the *P(eriodic) O(orbit) O(bstruction)*: if $p \in X$ and $n \in \mathbb{N}$ satisfy $T^n p = p$, then

$$A(n,p) = \prod_{i=0}^{n-1} A(T^i x) = \prod_{i=0}^{n-1} B(T^{i+1}x)B(T^i x)^{-1} = B(T^n p)B(p)^{-1} = e_{\mathcal{G}}.$$

The *Livšic problem* consists in determining whether the POO(A) condition is not only necessary but also sufficient for A being a coboundary. This terminology comes from the seminal work of Livšic [4], who proved that this is the case whenever $\mathcal{G}$ is Abelian, A is

Hölder-continuous and T is a topologically transitive hyperbolic diffeomorphism. Since then, many extensions of this classical result have been proposed. Perhaps the most relevant is Kalinin's recent version for $\mathcal{G} = \mathrm{GL}(d, \mathbb{C})$. In this Note, we address the Livšic problem for Hölder-continuous cocycles taking values in the group of germs of analytic diffeomorphisms. In the context of general diffeomorphisms, a positive answer to the Livšic problem is unclear, despite several results pointing in this direction whenever a certain localization property is satisfied. (See, for example, [2].)

To state our result, we denote by $\mathcal{G}erm_d$ the group of germs of local bi-holomorphisms of the complex space $\mathbb{C}^d$ fixing the origin. This may be identified to the group of holomorphic maps $F(Z) = A_1 Z + A_2 Z^2 + \ldots$ having positive convergence radious, with $A_1 \in \mathrm{GL}(d, \mathbb{C})$ (see §1.2 for the details).

Main Theorem. *Let $T : X \to X$ be a topologically transitive homeomorphism of a compact metric space X satisfying the closing property (see §1.1 for the details). Let $F : X \to \mathcal{G}erm_d$ be a Hölder-continuous function/cocycle (see §1.2 for a discussion on continuity issues). If F satisfies the POO condition, then there exists a Hölder-continuous function $H : X \to \mathcal{G}erm_d$ such that for all $x \in X$,*

$$F(x) = H(Tx) \circ H(x)^{-1}. \tag{2}$$

This theorem should be compared with [5], where the second-named author shows a KAM-type result for $\mathcal{G}erm_d$-valued cocycles over a minimal torus translation.

1.1 A remind on Livšic's theorem for complex valued cocycles

Let X be a compact metric space with normalized diameter (*i.e.*, $diam(X) = 1$). We say that a function $f : X \to \mathbb{C}$ is $(C, \alpha)-$Hölder-continuous for $C > 0$ and $\alpha \in (0, 1]$ if for every pair of points x, y in X,

$$|f(x) - f(y)| \leq C \, dist_X(x, y)^\alpha. \tag{3}$$

In the sequel, we will denote by $[f]_\alpha$ the smallest constant C for which f is $(C, \alpha)-$Hölder-continuous. The next two results are straightforward.

Lemma 1 *If f vanishes at some point of X, then $\|f\| := \sup_{x \in X} |f(x)| \leq [f]_\alpha$.* $\blacksquare$

Lemma 2 *Let $f, g : X \to \mathbb{C}$ be two α-Hölder-continuous functions. Then the functions $f + g$ and fg are $\alpha-$Hölder-continuous, and*

1. $[f + g]_\alpha \leq [f]_\alpha + [g]_\alpha$.
2. $[fg]_\alpha \leq [f]_\alpha \|g\| + [g]_\alpha \|f\|$. $\blacksquare$

Let $T\colon X \to X$ be a homeomorphism and let x, y be points of X. We say that the orbit segments $x, Tx, \ldots, T^k x$ and $y, Ty, \ldots, T^k y$ are *exponentially δ-close with exponent* $\lambda > 0$ if for every $j = 0, \ldots, k$,

$$dist_X(T^j x, T^j y) \leq \delta e^{-\lambda \min\{j, k-j\}}.$$

We say that T satisfies the *closing property* if there exist $c, \lambda, \delta_0 > 0$ such that for every $x \in X$ and $k \in \mathbb{N}$ so that $dist_X(x, T^k x) < \delta_0$, there exists a point $p \in X$ with $T^k p = p$ so that letting $\delta := c\, dist_X(x, T^k x)$, the orbit segments $x, Tx, \ldots, T^k x$ and $p, Tp, \ldots, T^k p$ are exponentially δ-close with exponent λ and there exists a point $y \in X$ such that for every $j = 0, \ldots, k$,

$$dist_X(T^j p, T^j y) \leq \delta e^{-\lambda j} \quad \text{and} \quad dist_X(T^j y, T^j x) \leq \delta e^{-\lambda(n-j)}.$$

Important examples of maps satisfying the closing property are hyperbolic diffeomorphisms of compact manifolds.

In this work, we will use two versions of the Livšic result. The first of these (*c.f.,* Theorem 3) is the original and seminal Livšic theorem for complex valued cocycles. This theorem will be used in an iterative scheme for which having estimates for the solutions of cohomological equations will be relevant (see Corollary 4). For this reason, we review the proof and we record certain crucial estimates. The second version (extension) of the Livšic result we will use (*c.f.*, Theorem 5) corresponds to a recent and remarkable theorem by B. Kalinin, who proves the Livšic theorem for matrix-valued cocycles (satisfying no localization condition).

Theorem 3 (Livšic, see [4]) *Let $T : X \to X$ be a topologically transitive homeomorphism of a compact metric space X satisfying the closing property. Let $\psi : X \to \mathbb{C}$ be an $\alpha-$Hölder-continuous function for which the POO holds, that is, for every point $p \in X$ and $k \geq 1$ such that $T^k p = p$, one has $\sum_{j=0}^{k-1} \psi(T^j p) = 0$. Then there exists an $\alpha-$Hölder-continuous function $\phi : X \to \mathbb{C}$ that is a solution to the cohomological equation*

$$\phi \circ T - \phi = \psi.$$

Proof. Let $x_0 \in X$ be such that $\overline{\{T^n x_0\}_{n\in\mathbb{N}}} = X$. We define ϕ by letting $\phi(x_0) := 0$ and $\phi(T^n x_0) := \sum_{j=0}^{n-1} \psi(T^j x_0)$. We next check that ϕ is $\alpha-$Hölder-continuous on $\{T^n x_0\}_{n\in\mathbb{N}}$. Let $n > m$. There are two cases to consider:

- Assume that $dist_X(T^m x_0, T^n x_0) < \delta_0$. Then there exists a point $p \in X$ satisfying $T^{n-m} p = p$ and such that for every $j = 0, \ldots, n-m$,

  $$dist_X(T^j(T^m x_0), T^j p) \leq c\, dist_X(T^n x_0, T^m x_0) e^{-\lambda \min\{j, n-m-j\}}.$$

  This yields

$$
\begin{aligned}
|\phi(T^n x_0) - \phi(T^m x_0)| &= \left| \sum_{j=0}^{n-m-1} \psi(T^{m+j} x_0) \right| \\
&= \left| \sum_{j=0}^{n-m-1} \big(\psi(T^{m+j} x_0) - \psi(T^j p) \big) + \sum_{j=0}^{n-m-1} \psi(T^j p) \right| \\
&\leq \sum_{j=0}^{n-m-1} \left| \psi(T^{m+j} x_0) - \psi(T^j p) \right| \\
&\leq \sum_{j=0}^{n-m-1} [\psi]_\alpha \, dist_X(T^{m+j} x_0, T^j p)^\alpha \\
&\leq \sum_{j=0}^{n-m-1} c^\alpha [\psi]_\alpha \, dist_X(T^n x_0, T^m x_0)^\alpha \, e^{-\lambda\alpha \min\{j, n-m-j\}} \\
&\leq \frac{2\, c^\alpha [\psi]_\alpha}{1 - e^{-\lambda\alpha}} \, dist_X(T^n x_0, T^m x_0)^\alpha.
\end{aligned}
$$

- Assume that $dist_X(T^n x_0, T^m x_0) \geq \delta_0$. Since x_0 has dense orbit and X is compact, there exists $N \in \mathbb{N}$, depending only on X, T, and δ_0, such that $\{x_0, Tx_0, \ldots, T^N x_0\}$ is a δ_0-dense set in X. For $n - m \leq N$, one easily shows that

$$|\phi(T^n x_0) - \phi(T^m x_0)| \leq N \|\psi\|.$$

For $n - m > N$, there exist r, s in $\{0, 1, \ldots, N\}$ such that $dist_X(T^s x_0, T^n x_0) \leq \delta_0$ and $dist_X(T^r x_0, T^m x_0) \leq \delta_0$. Using the preceding case, this yields

$$
\begin{aligned}
|\phi(T^n x_0) - \phi(T^m x_0)| &\leq |\phi(T^n x_0) - \phi(T^s x_0)| + |\phi(T^m x_0) - \phi(T^r x_0)| + |\phi(T^s x_0) - \phi(T^r x_0)| \\
&\leq \frac{4[\psi]_\alpha c^\alpha}{1 - e^{-\lambda\alpha}} \delta_0^\alpha + N\|\psi\| \\
&\leq \left(\frac{4[\psi]_\alpha c^\alpha}{1 - e^{-\lambda\alpha}} + \frac{N\|\psi\|}{\delta_0^\alpha} \right) dist_X(T^n x_0, T^m x_0)^\alpha. \quad \blacksquare
\end{aligned}
$$

A careful reading of the proof above yields useful estimates enclosed in the next

Corollary 4 *The solution ϕ to the cohomological equation is α-Hölder continuous, and there exists K depending only on T, X, and α such that $[\phi]_\alpha \leq K([\psi]_\alpha + \|\psi\|)$.* $\blacksquare$

Theorem 5 (Kalinin, see [3]) *Let T be a topologically transitive homeomorphism of a compact metric space X satisfying the closing property. Let $A : X \to GL(d, \mathbb{C})$ be an α-Hölder function for which the POO(A) holds. Then there exists an α-Hölder function $C : X \to GL(d, \mathbb{C})$ such that for all $x \in X$,*

$$A(x) = B(Tx) B(x)^{-1}.$$

1.2 The group $\mathcal{G}erm_d$

For $d \geq 1$, we introduce the following (classical) notation:

- $\mathbf{j} := (j_1, \ldots, j_d)$ is a positive integer lattice point, with $j_i \geq 0$ for every $1 \leq i \leq d$.
- $|\mathbf{j}| := j_1 + \cdots + j_d$.
- $\mathbf{j} \preceq \mathbf{k}$ if $j_i \leq k_i$ for every $1 \leq i \leq d$.
- $\mathbf{j} \prec \mathbf{k}$ if $\mathbf{j} \preceq \mathbf{k}$ and $j_{i_*} < k_{i_*}$ for some i_*.
- $Z = (z_1, z_2, \ldots, z_d)$ is a point in $\mathbb{C}^d$.
- $Z^{\mathbf{j}} := z_1^{j_1} z_2^{j_2} \cdots z_d^{j_d}$.

Then we can define a formal power series on $\mathbb{C}^d$ as $F(Z) := (F_1(Z), F_2(Z), \ldots, F_d(Z))$, where each $F_i(Z)$ has the form

$$F_i(Z) = \sum_{\mathbf{j} \geq 0} t^i_{\mathbf{j}} Z^{\mathbf{j}}$$

for some coefficients $t^i_{\mathbf{j}} \in \mathbb{C}$. This formal power series becomes an analytic map if there exists $R > 0$ such that $\limsup_{\mathbf{j}} |t^i_{\mathbf{j}}|^{\frac{1}{|\mathbf{j}|}} \leq \frac{1}{R}$ for every i. Indeed, in this case, each F_i is a convergent series on $D(0,R)^d$ (that is, for $Z = (z_1, \ldots, z_d)$ such that $|z_s| < R$ holds for every s).

Let $\mathcal{H}(d,R)$ be the set of continuous functions $F \colon \overline{D(0,R)^d} \to \mathbb{C}^d$ that are convergent power series in $D(0,R)^d$ and satisfy $F'(0) \in GL(d,\mathbb{C})$. We endow this complex vector space with the inner product

$$\langle F, G \rangle_R := \sum_i \left(\int_{\partial D(0,R)^d} F_i \overline{G_i} \, dZ \right).$$

The L^2-norm of an element $F \in \mathcal{H}(d,R)$ of the form $F_i(Z) = \sum_{|\mathbf{j}| \geq 0} t^i_{\mathbf{j}} Z^{\mathbf{j}}$ is

$$\|F\|_{2,R} := \langle F, F \rangle_R^{1/2} = \left(\sum_i \sum_{|\mathbf{j}| \geq 1} |t^i_{\mathbf{j}}|^2 R^{2|\mathbf{j}|} \right)^{1/2}.$$

We let $\mathcal{H}_0(d,R)$ be the subset of $\mathcal{H}(d,R)$ formed by those F satisfying $F(0) = 0$, and we let the set of *local holomorphic diffeomorphisms* of $\mathbb{C}^d$ be defined as

$$\mathcal{G}_d := \bigcup_{R > 0} \mathcal{H}_0(d,R).$$

On this set, we introduce the following equivalence relation: We say that F, G in $\mathcal{G}_d$ are equivalent if there exists a neighborhood of the origin on which F and G coincide. Under this identification, the set $\mathcal{G}_d$ becomes a group, that we call the *group of germs of analytic*

diffeomorphisms of $\mathbb{C}^d$ and we denote by $\mathcal{G}erm_d$.

Although we will not worry about giving a topology on $\mathcal{G}erm_d$, we will certainly need to consider maps from X to $\mathcal{G}erm_d$ that are "continuous" in some precise sense. Since X is compact, any reasonable definition should lead to functions that factor throughout an space $\mathcal{H}_0(d,R)$ for some positive R. Accordingly, given $C>0$, $\alpha\in(0,1]$, and $R>0$, a map $\Psi\colon X\to\mathcal{H}_0(d,R)$ will be said to be $(C,\alpha,R)-$Hölder-continuous if $\Psi(x)$ belongs to $\mathcal{H}_0(d,R)$ for every $x\in X$, and for every pair of points x,y in X,

$$\|\Psi(x)-\Psi(y)\|_{2,R}\leq C\; dist_X(x,y)^{\alpha}.$$

In terms of the coefficients of the power series, this condition reads as follows:

Lemma 6 *If $\Psi\colon X\to\mathcal{H}_0(d,R)$ is $(C,\alpha,R)-$Hölder and writes as*

$$\Psi_i(x)(Z)=\sum_{|\mathbf{j}|>0} t^i_{\mathbf{j}}(x)Z^{\mathbf{j}},$$

then each coefficient $t^i_{\mathbf{j}}:X\to\mathbb{C}$ is a $\left(\frac{C}{R^{|\mathbf{j}|}},\alpha\right)$-Hölder-continuous function.

Proof. The Hölder condition for Ψ yields

$$\left(\sum_i\sum_{|\mathbf{j}|\geq 1}|t^i_{\mathbf{j}}(x)-t^i_{\mathbf{j}}(y)|^2R^{2|\mathbf{j}|}\right)^{1/2}\leq C\; dist_X(x,y)^{\alpha},$$

which implies that

$$|t^i_{\mathbf{j}}(x)-t^i_{\mathbf{j}}(y)|^2\leq\frac{C^2}{R^{2|\mathbf{j}|}}dist_X(x,y)^{2\alpha}.\quad\blacksquare$$

In an opposite direction, given a list $\{t^i_{\mathbf{j}}:X\to\mathbb{C},\ \mathbf{j}\succeq 0,\ 1\leq i\leq d\}$ of continuous functions, we are interested in finding conditions ensuring that $F:=(F_1,\ldots,F_d)$ formally defined by $F_i(x)(Z):=\sum_{\mathbf{j}}t^i_{\mathbf{j}}(x)Z^{\mathbf{j}}$ represents a convergent power series lying in $\mathcal{H}_0(d,R)$ for some $R>0$.

Lemma 7 *Assume that each function $t^i_{\mathbf{j}}$ is a $\left(\frac{C}{R^{|\mathbf{j}|}},\alpha\right)$-Hölder-continuous function for some positive constants C,R. Assume also that each $t^i_{\mathbf{j}}$ vanishes at some point of X. Then for all $\delta<1$, the formal power series F_i above is convergent on $D(0,R)^d$, and $x\mapsto F(x)=(F_1(x),\ldots,F_d(x))$ is a $\left(O\left(\frac{\delta}{1-\delta}\right)^{1/2},\ \alpha\right)$-Hölder continuous map from X to $\mathcal{H}_0(d,\delta R)$.*

Proof. Since each $t^i_{\mathbf{j}}$ vanishes at some point of X, Lemma 1 gives $\|t^i_{\mathbf{j}}\|\leq\frac{C}{R^{|\mathbf{j}|}}$ for every $i,\mathbf{j}$. This implies that each F_i is a convergent power series on $D(0,R)^d$. Moreover, for all x,y in X,

$$
\begin{aligned}
\|F(x)-F(y)\|_{2,\delta R}^2 &= \sum_i \sum_{\mathbf{j}} |t_{\mathbf{j}}^i(x)-t_{\mathbf{j}}^i(y)|^2 (\delta R)^{2|\mathbf{j}|} \\
&\le \sum_i \sum_{\mathbf{j}} C^2 dist_X(x,y)^{2\alpha} \delta^{2|\mathbf{j}|} \\
&= d\, C^2 dist_X(x,y)^{2\alpha} \sum_{s=1}^{\infty} \sum_{|\mathbf{j}|=s} \delta^{2s} \\
&= d\, C^2 dist_X(x,y)^{2\alpha} \sum_{s=1}^{\infty} \frac{(s+d-1)!}{s!(d-1)!} \delta^{2s} \\
&= d\, C^2\, O\!\left(\frac{\delta}{1-\delta}\right) dist_X(x,y)^{2\alpha}. \quad \blacksquare
\end{aligned}
$$

The Faa di Bruno formula. We will need to consider compositions of power series in several complex variables. The following is a simplified formulation of the multivariate version by Constantine and Savits [1] of the famous Faa di Bruno formula:

Theorem 8 (see [1]) *Let* $A(Z) = \sum_{|\mathbf{j}|\ge 1} a_{\mathbf{j}} Z^{\mathbf{j}}$ *and* $B_i(Z) = \sum_{|\mathbf{j}|\ge 1} b_{\mathbf{j}}^i Z^j$, $1 \le i \le d$, *be formal power series in* d *variables. Then the power series*

$$C(Z) = A\left(B_1(Z), B_2(Z), \ldots, B_d(Z)\right) = \sum_{|\mathbf{j}|\ge 1} c_{\mathbf{j}} Z^{\mathbf{j}}$$

has coefficients

$$c_{\mathbf{j}_*} = \sum_{|\mathbf{j}|=\mathbf{1}} a_{\mathbf{j}} b_{\mathbf{j}_*}^{\mathbf{j}} + \sum_{1<|\mathbf{j}|,\ \mathbf{j}\le \mathbf{j}_*} a_{\mathbf{j}} P(\mathbf{j}_*, \mathbf{j})\{B\}, \tag{4}$$

where $P(\mathbf{j}_*, \mathbf{j})\{B\}$ *is polynomial in the variables* $\{b_{\mathbf{j}}^i\}_{\mathbf{j}<\mathbf{j}_*}^{1\le i\le d}$ *that is homogeneous of degree* $|\mathbf{j}|$ *and has positive integer coefficients.*

The Faa di Bruno formula is actually much more precise but requires hard notation. For instance, in the case $d=1$, one has

$$P(j_*, j)\{B\} = \sum_{r_1+\cdots+r_j = j_*} B_{r_1} \cdots B_{r_j}.$$

A generating function. Let us define $J : D(0,1)^d \to \mathbb{C}^d$ by the convergent power series

$$J_i(Z) = z_i - \sum_{|\mathbf{j}|>1} Z^{\mathbf{j}}.$$

Since $DJ(0) = id_{\mathbb{C}^d}$, there exists an analytic map G defined in a neighborhood of the origin in $\mathbb{C}^d$ such that $G(0)=0$ and

$$J \circ G(Z) = Z \quad \text{for every } Z \text{ in that neighborhood .} \tag{5}$$

In terms of power series, one can write

$$G_i(Z) = z_i + \sum_{|\mathbf{j}|>1} g^i_{\mathbf{j}} Z^{\mathbf{j}},$$

where the coefficients verify $|g^i_{\mathbf{j}}| < K^{|\mathbf{j}|-1}$ for some $K > 0$ and every $|\mathbf{j}| \succ 1$. Moreover, these coefficients satisfy a fundamental recurrence relation. Indeed, using $J \circ G(Z) = Z$ and the Faa di Bruno formula (4), one obtains

$$0 \;=\; g^i_{\mathbf{j}_*} \;-\; \sum_{1<|\mathbf{j}|,\ \mathbf{j}\leq \mathbf{j}_*} P(\mathbf{j}_*, \mathbf{j})\{G\}. \tag{6}$$

Recall that $P(\mathbf{j}_*, \mathbf{j})\{G\}$ depends only on the values of $g^s_{\tilde{\mathbf{j}}}$ for $\tilde{\mathbf{j}} \prec \mathbf{j}_*$ and every s. Hence, one can recursively compute $g^i_{\mathbf{j}_*}$ in terms of the previously defined $g^s_{\tilde{\mathbf{j}}}$.

For any $S > 0$, we consider $J_S : D(0, S^{-1})^d \to \mathbb{C}^d$ defined by $J_S(Z) := \frac{1}{S} J(SZ)$. When solving the equation $J_S \circ G_S(Z) = Z$, one gets a map $G_S = (G_{S,1}, \ldots, G_{S,d})$, where each $G_{S,i}(Z)$ has the form $G_{S,i}(Z) = z_i + \sum_{|\mathbf{j}|>1} g^i_{S,\mathbf{j}} Z^{\mathbf{j}}$ for certain coefficients $g^i_{S,\mathbf{j}}$ satisfying

$$g^i_{S,\mathbf{j}_*} = \sum_{1<|\mathbf{j}|,\ \mathbf{j}\leq \mathbf{j}_*} S^{|\mathbf{j}|-\mathbf{1}} P(\mathbf{j}_*, \mathbf{j})\{G_S\}. \tag{7}$$

Lemma 9 *Each coefficient $g^i_{S,\mathbf{j}}$ is a positive real number. Moreover, there exists a constant $\mathcal{R} = \mathcal{R}(S) > 0$ such that $g^i_{S,\mathbf{j}} \leq \mathcal{R}^{|\mathbf{j}|-1}$ for every $\mathbf{j}$.* ■

1.3 Proof of the Main Theorem

A first reduction. Let $F(x)(Z) = A_1(x)Z + \left(\sum_{|\mathbf{j}|>1} a^i_{\mathbf{j}}(x) Z^{\mathbf{j}}\right)_{1\leq i\leq d}$ be the power series expansion of the cocycle viewed as a $(C, \alpha, R)-$Hölder-continuous function $\Psi : X \to \mathcal{H}_0(d, R)$. The map $x \mapsto A_1(x) \in GL(d, \mathbb{C})$ is an α-Hölder-continuous function. Since POO(F) holds, we must have

$$\prod_{j=0}^{n-1} A_1(T^j p) \;=\; \frac{\partial}{\partial Z} F(T^{n-1}p) \circ \cdots \circ F(p)\Big|_{Z=0} \;=\; id_{\mathbb{C}^d}$$

for every $p \in X$ and $n \in \mathbb{N}$ such that $T^n p = p$. In other words, the $GL(d, \mathbb{C})$-valued cocycle A_1 satisfies the POO. By Kalinin's version of the Livšic theorem, there exists an α-Hölder-continuous function $H_1 : X \to GL(d, \mathbb{C})$ such that $A_1(x) = H_1(Tx) H_1(x)^{-1}$ for all $x \in X$. Consequently, the $\mathcal{G}erm_d$-valued cocycle $H_1(x)(Z) := H_1(x)Z$ conjugates F to a cocycle of the form

$$(x, Z) \longmapsto \left(Tx, Z + \left(\sum_{|\mathbf{j}|>1} a^i_{\mathbf{j}}(x) Z^{\mathbf{j}} \right)_{1\leq i\leq d} \right).$$

Thus, we can assume that $A_1(x) = id_{\mathbb{C}^d}$ for all $x \in X$.

An iterative procedure. We look for a map $H\colon X \to \mathcal{G}erm_d$ solving the cohomological equation (2) having the form $H(x)(Z) = Z + \left(\sum_{|\mathbf{j}|>1} h^i_{\mathbf{j}}(x) Z^{\mathbf{j}}\right)_{1\le i\le d}$. Notice that this equation may be written as $F(x)\circ H(x) = H(Tx)$. Applying the Faa di Bruno formula (4) to the left-side expression, one concludes that each coefficient $h^i_{\mathbf{j}}$ can be defined recursively as the solution of a cohomological equation for a $\mathbb{C}$-valued data:

$$(ec^i_{\mathbf{j}_*}) \qquad h^i_{\mathbf{j}_*}(Tx) - h^i_{\mathbf{j}_*}(x) \;=\; \textstyle\sum_{1<|\mathbf{j}|,\ \mathbf{j}\le\mathbf{j}_*} a^i_{\mathbf{j}}(x) P(\mathbf{j}_*, \mathbf{j})\{H\}(x).$$

A necessary condition for the existence of the coefficient $h^i_{\mathbf{j}_*}$ is that the POO condition holds for the function

$$R^i_{\mathbf{j}_*} := \sum_{1<|\mathbf{j}|,\ \mathbf{j}\le\mathbf{j}_*} a^i_{\mathbf{j}} P(\mathbf{j}_*, \mathbf{j})\{H\}. \tag{8}$$

Lemma 10 *Each $R^i_{\mathbf{j}_*}$, with $i, |\mathbf{j}_*| \succ 1$, is a well-defined α-Hölder-continuous function for which the POO holds. As a consequence, given any $x_0 \in X$, the equation $(ec^i_{\mathbf{j}_*})$ has an α-Hölder-continuous solution $h^i_{\mathbf{j}_*}$ vanishing at x_0.*

Proof. Suppose that the conclusion of the lemma holds for every $\mathbf{j}$ such that $|\mathbf{j}| < k$, and let us consider the case where $\mathbf{j} = k$. Using the explicit formula (8), Lemma 2 shows that the function $R^i_{\mathbf{j}_*}$ is α-Hölder-continuous. Consider the continuous $\mathcal{G}erm_d$-valued function

$$H_{<k} : x \mapsto Z + \left(\sum_{|\mathbf{j}|<k} h^i_{\mathbf{j}}(x) Z^{\mathbf{j}}\right)_{1\le i\le d}.$$

An easy computation shows that $\tilde{F}(x) := H_{<k}(Tx) \circ F(x) \circ H_{<k}(x)^{-1}$ has the form

$$\tilde{F}(x)(Z) = Z + \left(\sum_{|\mathbf{j}|=k} R^i_{\mathbf{j}}(x) Z^{\mathbf{j}} + \sum_{|\mathbf{j}|>k} \tilde{a}^i_{\mathbf{j}}(x) Z^{\mathbf{j}}\right)_{1\le i\le d}$$

for some Hölder-continuous functions $\tilde{a}^i_{\mathbf{j}} : X \to \mathbb{C}$. Moreover, for any $x \in X$ and $m \in \mathbb{N}$, one has

$$\tilde{F}(T^{m-1}x) \circ \cdots \circ \tilde{F}(x)(Z) = Z + \left(\sum_{|\mathbf{j}|=k} \left(\sum_{v=0}^{m-1} R^i_{\mathbf{j}}(T^v x)\right) Z^{\mathbf{j}} + \mathcal{O}(|Z|^{k+1})\right)_{1\le i\le d}.$$

Since $\tilde{F}$ is conjugated to F, the POO($\tilde{F}$) holds. By the previous equality, this implies that for all $p \in X$ and $n \in \mathbb{N}$ such that $T^n p = p$, one has $\sum_{v=0}^{n-1} R^i_{\mathbf{j}}(T^v x) = 0$. Therefore, the POO($R^i_{\mathbf{j}}$) holds, which allows applying Livsic's theorem to ensure the existence of an α-Hölder-continuous solution to $(ec^i_{\mathbf{j}_*})$. Finally, by adding a constant if necessary, we may assume that this solution vanishes at x_0. ∎

To prove that the (up to now) formal map H is a genuine local diffeomorphism (that is, each formal power series $Z \mapsto z_i + \sum_{|\mathbf{j}|>1} h^i_{\mathbf{j}}(x) Z^{\mathbf{j}}$ is convergent in a certain (uniform) neighborhood of the origin), we will need to estimate the growth of the α-Hölder constant of the coefficients $h^i_{\mathbf{j}}$. Indeed, if we show that this growth is at most exponential, then Lemma 7 will apply, thus concluding the proof of the Main Theorem. To get the desired control, we will use the *majorant series method* introduced by Siegel in his treatement [6] of the linearization theorem for holomorphic germs with Diophantine rotation number (see also [7] for the higher-dimensional case).

Lemma 11 *There exists $S > 0$ such that*

$$[h^i_{\mathbf{j}}]_\alpha \leq g^i_{S,\mathbf{j}}$$

for every $\mathbf{j}, i$, where $h^i_{S,\mathbf{j}}$ is defined as in (7). Consequently, $\|h^i_{\mathbf{j}}\|$ grows at most exponentially.

Proof. Since F takes values on some $\mathcal{H}_0(d,R)$ and is a α-Hölder function, there exists $\kappa > 0$ such that

$$\|a^i_{\mathbf{j}}\| \leq \kappa^{|\mathbf{j}|} \quad \text{and} \quad [a^i_{\mathbf{j}}]_\alpha \leq \kappa^{|\mathbf{j}|}.$$

Assume that $[h^i_{\mathbf{j}}]_\alpha \leq g^i_{S,\mathbf{j}}$ for every $\mathbf{j} \preceq \mathbf{j}_*$. Since $h^i_{\mathbf{j}}$ vanishes at x_0 (except for $|\mathbf{j}| = \mathbf{1}$, for which $h^i_{\mathbf{j}} \equiv 1$), we also have $\|h^i_{\mathbf{j}}\| \leq g^i_{S,\mathbf{j}}$ for every $\mathbf{j} \preceq \mathbf{j}_*$. Moreover, since $P(\mathbf{j}_*, \mathbf{j})\{H\}$ is an homogeneous polynomial in $\{h^s_{\tilde{\mathbf{j}}}\}^{1 \leq s \leq d}_{\tilde{\mathbf{j}} < \mathbf{j}_*}$ with positive coefficients,

$$\|P(\mathbf{j}_*, \mathbf{j})\{H\}\| \leq P(\mathbf{j}_*, \mathbf{j})\{\|H\|\} \leq P(\mathbf{j}_*, \mathbf{j})\{G_S\}.$$

Except for $|\mathbf{j}| = \mathbf{1}$ (for which $h^i_{\mathbf{j}} \equiv 1$), every $h^i_{\mathbf{j}}$ vanishes at x_0. Therefore, by Lemma 2,

$$[P(\mathbf{j}_*, \mathbf{j})\{H\}]_\alpha \leq 2^{|\mathbf{j}|-\mathbf{1}} P(\mathbf{j}_*, \mathbf{j})\{G_S\}.$$

The fundamental estimate of Corollary 4 then yields

$$\begin{aligned}
[h^i_{\mathbf{j}_*}]_\alpha &\leq K \left(\left[\sum_{\mathbf{j} \leq \mathbf{j}_*} a^i_{\mathbf{j}} P(\mathbf{j}_*, \mathbf{j})\{H\} \right]_\alpha + \left\| \sum_{\mathbf{j} \leq \mathbf{j}_*} a^i_{\mathbf{j}} P(\mathbf{j}_*, \mathbf{j})\{H\} \right\| \right) \\
&\leq K \left(\sum_{\mathbf{j} \leq \mathbf{j}_*} \|a^i_{\mathbf{j}}\| [P(\mathbf{j}_*, \mathbf{j})\{H\}]_\alpha + \sum_{\mathbf{j} \leq \mathbf{j}_*} [a^i_{\mathbf{j}}]_\alpha \|P(\mathbf{j}_*, \mathbf{j})\{H\}\| + \sum_{\mathbf{j} \leq \mathbf{j}_*} \|a^i_{\mathbf{j}}\| \|P(\mathbf{j}_*, \mathbf{j})\{H\}\| \right) \\
&\leq \sum_{\mathbf{j} \leq \mathbf{j}_*} K \left((2\kappa)^{|\mathbf{j}|} + 2\kappa^{|\mathbf{j}|} \right) P(\mathbf{j}_*, \mathbf{j})\{G_S\} \\
&< g^i_{S,\mathbf{j}_*},
\end{aligned}$$

where the last inequality holds by taking $S \gg 2K\kappa$. ∎

Acknowledgments. Both authors were funded by the Math-AMSUD Project DySET. Mario Ponce was also funded by the Fondecyt Grant 11090003.

References

[1] G.M. Constantine & T.H. Savits. A multivariate Faa di Bruno formula with applications. *Transactions of Amer. Math. Soc.* **348**, no. **2** (1996), 503-520.

[2] R. de la Llave & A. Windsor. Livšic theorems for non-commutative groups including diffeomorphism groups and results on the existence of conformal structures for Anosov systems. *Ergodic Theory and Dynam. Systems* **30**, no. **4** (2010), 1055-110.

[3] B. Kalinin. Livšic theorem for matrix cocycles. *Annals of Math.* **173**, no **2** (2011), 1025-1042.

[4] A.N. Livšic. Cohomology of dynamical systems. *Math. USSR Izvestija* **6** (1972), 1278-1301.

[5] M. Ponce. Towards a semi-local study of parabolic invariant curves for fibered holomorphic maps. To appear in *Ergodic Theory and Dynam. Systems.*

[6] C.L. Siegel. Iteration of analytic functions. *Annals of Math.* **43**, no **2** (1942), 607-612.

[7] S. Sternberg. Infinite Lie groups and the formal aspects of Dynamical Systems. *Journal of Mathematics and Mechanics* **10**, no. **3** (1961), 451-474.

Andrés Navas
Dpto de Matemática y C.C., USACH
Alameda 3363, Estación Central, Santiago, Chile
E-mail: andres.navas@usach.cl

Mario Ponce
Facultad de Matemáticas, PUC
Casilla 306, Santiago 22, Chile
E-mail: mponcea@mat.puc.cl